Operator Product Expansion and
Zero Mode Structure in logarithmic CFT

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Abstract: The generic structure of 1-, 2- and 3-point functions of fields residing in indecomposable representations of arbitrary rank are given. These in turn determine the structure of the operator product expansion in logarithmic conformal field theory. The crucial role of zero modes is discussed in some detail.

1   Introduction

During the last few years, logarithmic conformal field theory (LCFT) has been established as a well-defined variety of conformal field theories in two dimensions. The concept was considered in its own right first by Gurarie [11], Since then, a large amount of work has appeared, see the reviews [8, 10] and references therein. The defining feature of a LCFT is the occurence of indecomposable representations which, in turn, may lead to logarithmically diverging correlation functions. Thus, in the standard example of a LCFT a primary field $\phi(z)$ of conformal weight h has a so-called logarithmic partner field ψ with the characteristic properties

$$\langle\phi(z)\phi(0)\rangle = 0\,, \quad \langle\phi(z)\psi(0)\rangle = Az^{-2h}\,, \quad \langle\psi(z)\psi(0)\rangle = z^{-2h}\left(B - 2A\log(z)\right)\,. \quad (1)$$

To this corresponds the fact that the highest weigth state $|h\rangle$ associated to the primary field ϕ is the ground state of an irreducible representation which, however, is part of a larger, indecomposable, representation created from $|\tilde{h}\rangle$, the state associated to ψ. The conformal weight is the eigenvalue under the action of L_0, the zero mode of the Virasoro algebra, which in such LCFTs cannot be diagonalized. Instead, we have

$$L_0|h\rangle = h|h\rangle\,, \quad L_0|\tilde{h}\rangle = h|\tilde{h}\rangle + |h\rangle\,. \quad (2)$$

Thus, the two states $|h\rangle$ and $|\tilde{h}\rangle$ span a Jordan cell of rank two with respect to L_0. As can be guessed from eq. (1), there must exist a zero mode which is responsible for the vanishing of the 2-pt function of the primary field. Another characteristic fact in LCFT is the existence of at least one field, which is a perfect primary field, but whose operator product expansion (OPE) with itself produces a logarithmic field. Such fields

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μ are called pre-logarithmic fields [14]. This is important, since in many cases, the pre-logarithmic fields arise naturally forcing us then to include the logarithmic fiels as well into the operator algebra. Note that this implies that the fusion product of two irreducible representations is not necessarily completely reducible into irreducible representations. In fact, we know today quite a few LCFTs, where precisely this is the case, such as ghost systems [16], WZW models at level zero or at fractional level such as $\widehat{SU(2)}_{-4/3}$ [9, 15], WZW models of supergroups such as $GL(1,1)$ [20] or certain supersymmetric $c = 0$ theories such as $OSP(2n|2n)$ or $CP(n|n)$ [12, 18]. Finally, many LCFTs are generated from free anticommuting fields such as the symplectic fermions [13]. Such LCFTs have an interesting fermionic structure where logarithms may also arise in correlation functions involving spin zero anticommuting fields. This is in contrast to free bosons, which typically do not directly appear in the conformal field theory, but only in form of derivatives and exponentials of themselves.

In these notes, we generalize LCFT to the case of Jordan cells of arbitrary rank, but we will restrict ourselves to the Virasoro algebra as the chiral symmetry algebra to keep things simple. With some mild assumptions, the generic form of 1-, 2- and 3-pt functions can be given such that only the structure constants remain as free parameters. From this, the general structure of the OPE as well as some sort of selection rules that a general correlation function may be non-zero, are derived. The crucial role of zero modes, in particular in the case of a LCFT generated from fermionic fields, is emphasized. The results presented here, together with proofs and further details, can be found in [7, 16]. The computation of 4-pt and higher-point functions in the LCFT case is, unfortunatley, more complicated. The interested reader might consult [5, 6] for some discussion on this issue.

2 1-, 2- and 3-pt functions

Let r denote the rank of the Jordan cells we consider. One can show, that in LCFTs with Jordan cells with respect to (at least) the L_0 mode, the $h = 0$ sector necessarily must carry such a Jordan cell structure. Furthermore, its rank defines the maximal possible rank of all Jordan cells. Thus, without loss of generality, we can assume that the rank of all Jordan cells is equal to r, other cases can easily be obtained by setting certain structure constants to zero. Each Jordan cell contains one proper highest weight state giving rise to one proper irreducible subrepresentation. We will label this state for a Jordan cell with conformal weight h by $|h; 0\rangle$. We choose a basis in the Jordan cell with states $|h; k\rangle$, $k = 0, \ldots, r - 1$, such that eq. (2) is replaced by

$$L_0|h; k\rangle = h|h; k\rangle + |h; k - 1\rangle \ \ \text{for} \ \ k = 1, \ldots, r - 1, \ \ \ \ L_0|h; 0\rangle = h|h; 0\rangle . \quad (3)$$

The corresponding fields will be denoted $\Psi_{(h;k)}$. Although the OPE of two primary fields might produce logarithmic fields, we will further assume, that primary fields *which are members of Jordan cells* are proper primaries in the sense that OPEs among them only yield again primaries.

As discussed by Rohsiepe [19], the possible structures of indecomposable representations with respect to the Virasoro algebra are surprisingly rich. Besides the defining condition eq. (3), further conditions have to be employed to fix the structure. The simplest case is defined via the additional requirement

$$L_1|h; k\rangle = 0 , \ \ \ \ 0 \leq k < r . \quad (4)$$

This condition means that all fields spanning the Jordan cell are quasi-primary. It will be our starting point in the following. Under these assumptions, as shown in [4], the action of the Virasoro modes receives an additional non-diagonal term. The off-diagonal action is defined via $\hat{\delta}_{h_i}\Psi_{(h_j;k_j)}(z) = \delta_{ij}\Psi_{(h_j;k_j-1)}(z)$ for $k_j > 0$ and $\hat{\delta}_{h_i}\Psi_{(h_j;0)}(z) = 0$. Thus,

$$L_n\langle\Psi_{(h_1;k_1)}(z_1)\ldots\Psi_{(h_n;k_n)}(z_n)\rangle = \sum_i z_i^n \left[z_i\partial_i + (n+1)(h_i + \hat{\delta}_{h_i})\right]\langle\Psi_{(h_1;k_1)}(z_1)\ldots\Psi_{(h_n;k_n)}(z_n)\rangle \tag{5}$$

for $n \in \mathbb{Z}$. Only the generators L_{-1}, L_0, and L_1 of the Möbius group admit globally valid conservation laws, which usually are expressed in terms of the so-called conformal Ward identities

$$0 = \begin{cases} L_{-1}G(z_1,\ldots z_n) &= \sum_i \partial_i G(z_1,\ldots z_n)\,, \\ L_0 G(z_1,\ldots z_n) &= \sum_i (z_i\partial_i + h_i + \hat{\delta}_{h_i})G(z_1,\ldots z_n)\,, \\ L_1 G(z_1,\ldots z_n) &= \sum_i (z_i^2\partial_i + 2z_i[h_i + \hat{\delta}_{h_i}])G(z_1,\ldots z_n)\,, \end{cases} \tag{6}$$

where $G(z_1,\ldots z_n)$ denotes an arbitrary n-point function $\langle\Psi_{(h_1;k_1)}(z_1)\ldots\Psi_{(h_n;k_n)}(z_n)\rangle$ of primary fields and/or their logarithmic partner fields. Here, we already have written down the Ward identities in the form valid for proper Jordan cells in logarithmic conformal field theories. Note that these are now inhomogeneous equations. In principle, this is all one needs to compute the generic form of all n-pt functions, $n \leq 3$ upto structure constants. Thus, using freedom of scaling the fields, the 1-pt functions turn out to be

$$\langle\Psi_{(h;k)}\rangle = \delta_{h,0}\delta_{k,r-1}\,. \tag{7}$$

The 2-pt and 3-pt functions can be written in a rather compact form by noting that derivaties of z^h with respect to h yields a logarithm. The structure constants depend on both, the conformal weights as well as the *total level* within the Jordan cells. One obtains

$$\langle\Psi_{(h;k)}(z)\Psi_{(h',k')}(z')\rangle = \sum_{j=r-1}^{k+k'} D_{(h;j)}\,\delta_{h,h'} \sum_{\substack{0\leq i\leq k, 0\leq i'\leq k' \\ i+i'=k+k'-j}} \frac{1}{i!i'!}(\partial_h)^i(\partial_{h'})^{i'}(z-z')^{-h-h'} \tag{8}$$

for the 2-pt functions, and for the 3-pt functions analogously

$$\langle\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2,k_2)}(z_2)\Psi_{(h_3;k_3)}(z_3)\rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1,h_2,h_3;j)} \sum_{\substack{0\leq i_l\leq k_l, l=1,2,3 \\ i_1+i_2+i_3=k_1+k_2+k_3-j}} \frac{1}{i_1!i_2!i_3!}$$
$$\times (\partial_{h_1})^{i_1}(\partial_{h_2})^{i_2}(\partial_{h_3})^{i_3}(z_{12})^{h_3-h_1-h_2}(z_{13})^{h_2-h_1-h_3}(z_{23})^{h_1-h_2-h_3}\,. \tag{9}$$

3   OPEs

It is now a simple matter to write down the generic form of OPEs. In essence, we have to raise one index of the 3-pt structure constants with the help of the inverse of the 2-pt structure constants, *i.e.* the propagators. Now, in the LCFT case, we have matrices instead, namely

$$(D_{h,h'})_{k,k'} \equiv \delta_{h,h'}\langle\Psi_{(h;k)}(z_2)\Psi_{(h',k')}(z_3)\rangle\,, \tag{10}$$

which is an upper triangular matrix and thus invertible, and

$$\left(C_{(h_1;k1),h_2,h_3}\right)_{k_2,k_3} \equiv \langle\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2,k_2)}(z_2)\Psi_{(h_3;k_3)}(z_3)\rangle\,. \tag{11}$$

The OPE is then given by the expression

$$\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2) = \lim_{z_1 \to z_2} \sum_{(h_3;k_3)} \sum_k \left(C_{(h_1;k_1),h_2,h_3}\right)_{k_2,k} \left((D_{h_3,h_3})^{-1}\right)^{k,k_3} \Psi_{(h_3;k_3)}(z_2),$$
(12)

where the limit means that we have to replace z_{13} in the result by z_{23} which, in fact, will cancel all dependency on z_3. In this form, the OPE does not obey a bound such as $k_3 \leq k_1 + k_2$ for the so-called J-levels within the Jordan blocks. For example, pre-logarithmic fields are good primary fields, such that $k_1 = k_2 = 0$, while there appears a term with $k_3 = 1$ on the right hand side.

A better bound is given by the *zero mode content* of the fields. This means the following: The basic fields of the conformal field theory might contain a certain number of zero modes $\theta_0^{(\alpha)}$ such that $\langle 0|\theta_0^{(\alpha)} = \theta_0^{(\alpha)}|0\rangle = 0$. These modes will come with canonical conjugate modes $\xi_{(\alpha)}$, which are creators to the right as well as to the left. Thus, the zero mode content $Z_0(\Psi)$ of a field Ψ is defined as the total number of $\xi_{(\alpha)}$ modes in its mode expansion, expressed in the modes of the basic fields. If the basic fields are anticommuting fermions, we will have anticommuting pairs $\xi_{(\alpha)}^\pm$ instead such that we can define zero mode contents $Z_+(\Psi)$ and $Z_-(\Psi)$ separately with $Z_0 = Z_+ + Z_-$. Explicitly known examples of LCFTs do have realizations in fermionic free fields, and it turns out that the definition above can be extended to pre-logarithmic fields in a consistent way by assigning them fractional values Z_+ and Z_- such that always $Z_0 \in \mathbb{Z}$. In fact, a large class of LCFTs can be constructed from ordinary conformal field theories by introducing additional zero modes accompanied with a suitable deformation of the Virasoro modes, see [1] for details. The zero mode content does now provide a bound for OPEs, namely

$$Z_0(\Psi_{(h_3;k_3)}) \leq Z_0(\Psi_{(h_1;k_1)}) + Z_0(\Psi_{(h_2;k_2)}).$$
(13)

One of the best known examples for a LCFT is the $c = -2$ ghost system, written in terms of two anticommuting spin zero fields $\theta^\pm(z)$. The mode expansion reads

$$\theta^\pm(z) = \xi^\pm + \theta_0^\pm \log(z) + \sum_{n \neq 0} \theta_n^\pm z^{-n},$$
(14)

where the modes $\xi^\pm$ are the creator zero modes, while the modes $\theta_0^\pm$ are the annihilator zero modes, satisfying $\{\xi^\pm, \theta_0^\mp\} = 1$, $\{\theta_n^+, \theta_m^-\} = \frac{1}{n}\delta_{n+m,0}$. We get back the original bc ghost system by setting $c(z) = \theta^-(z)|_{\theta_0^- = 0}$ and $b(z) = \partial_z\theta^+(z)$. Thus, the pair θ_0^- and ξ^+ of zero modes is absent in the bc system, and so is the logarithmic partner of the identity field, $\Psi_{(h=0;1)}(z) = :\theta^+\theta^-:(z)$ with state $|0;1\rangle = \xi^+\xi^-|0;0\rangle$, where $|0;0\rangle = |0\rangle$. Thus, $Z_0(\Psi_{(0;0)}) = 2$, $Z_\pm(\Psi_{(0;0)}) = 1$, while the basic fermionic fields obey $Z_0(\theta^\pm) = 1$, $Z_\pm(\theta^\pm) = 1$ and $Z_\mp(\theta^\pm) = 0$. This theory possesses a pre-logarithmic field μ of conformal weight $h = -1/8$ with OPE $\mu(z)\mu(0) \sim z^{1/4}\left(\Psi_{(0;1)}(0) - 2\log(z)\Psi_{(0;0)}(0)\right)A + z^{1/4}\Psi_{(0;0)}(0)B$. To make everything consistent, one assigns $Z_\pm(\mu) = 1/2$. We mention for completeness, that the excited twist field σ with conformal weight $h = 3/8$ has to be assigned the values $Z_+(\sigma) = 3/2$ and $Z_-(\sigma) = -1/2$ or vice versa.

4 Zero mode content

Let us briefly consider a much less trivial example, the ghost system with $c = -26$, made out of a pair of anticommuting fiels of spin 2 and -1, respectivelyi [16]. In general, the

$(j, 1-j)$ ghost system possesses $2j-1$ zero modes $b_{j-1}, b_{j-2}, \ldots, b_{1-j}$. The stress energy tensor reads

$$T_{bc} = -j{:}b(\partial c){:} + (1-j){:}(\partial b)c{:}\,. \tag{15}$$

Using a slight generalization of the deformation technique of [1], additional zero modes can be introduced by a modification, shown here for the $j=2$ case,

$$T_{\log}(z) = T_{bc}(z) + A\theta_1^- \partial b(z) + B\theta_0^- z^{-1}\partial(z^2 b(z)) + A\theta_{-1}^- z^{-2}\partial(z^4 b(z))\,. \tag{16}$$

These additional zero modes can be thought of as modes of $h=-1$ fields $\theta^\pm$ with expansion

$$\theta^\pm(z) = \xi_{-1}^\pm z^2 + \xi_0^\pm z + \xi_{+1}^\pm + \theta_{-1}^\pm \frac{z^2}{2}(\log(z) - \frac{3}{2}) + \theta_0^\pm z(\log(z) - 1) + \theta_{+1}^\pm \log(z) + \sum_{|n|>1} \theta_n^\pm \frac{z^{-n+1}}{1-n}\,. \tag{17}$$

Again, $b(z) = \partial^{2j-1}\theta^+(z)$ and $c(z) = \theta^-(z)|_{\theta_{j-1}^- = \theta_{j-2}^- = \ldots = \theta_{1-j}^- = 0}$ such that the $\theta^\pm$ fields have twice as many zero modes as the original bc system, and $\{\xi_i^\pm, \theta_{-i}^\mp\} = \pm(-1)^{i+1}$. Although the modes of the modified stress energy tensor satisfy the Virasoro algebra, they do not act consistently on the space of states, e.g. $L_0|\xi_{-1}^+\rangle = 0$.

However, as explained in [16], the doubling of the zero modes is not completely articifial, but does naturally imply that the conformal field theory (CFT) now lives on a hyperellptic Riemann surface, viewed as a double covering of the complex plane or Riemann sphere. Thus, we actually have a CFT on each of the sheets, such that the full CFT is the tensor product of the individual ones with $T_{\log} = T_{\log}^{(1)} + T_{\log}^{(2)}$ such that $[T_{\log}^{(1)}, T_{\log}^{(2)}] = 0$. In fact, this is possible and yields a consistent CFT provided we identify the zero modes on the different sheets with each other as $\theta_i^{(1),\pm} = -\theta_i^{(2),\mp}$ and $\xi_i^{(1),\pm} = \xi_i^{(2),\mp}$ for $i = -1, 0, 1$. This yields the Virasoro algebra for $T_{\log}$ with total central charge $c = -52$ and a correct action of its modes on the space of states. The resulting theory possesses indecomposable representations despite the fact that the action of L_0 remains diagonal. The construction generalizes to other ghost systems, but is is not yet clear, how the construction works for higher ramified covering with more than two sheets.

This shows that zero modes are at the heart of LCFTs. Furthermore, the zero mode content provides strong conditions on whether correlation functions can actually be non-zero. It appears that fields $\Psi_{(h;k)}$ forming Jordan blocks have a well defined *even* zero mode content $Z_0 = Z_+ + Z_-$ with $Z_+ = Z_-$. Fermionic fields, in turn, are characterized by $Z_+ \neq Z_-$ but still satisfy $Z_\pm \in \mathbb{Z}$. These fields are denoted by $\Theta_{(h;k^+,k^+)}$. There are no examples known where such fields do form indecomposable structures, but this is only due to the fact that no LCFTs with a sufficiently high number of genuine zero modes have been explicitly examined yet. Finally, pre-logarithmic, or more generally, twist fields, have fractional zero mode contents, $Z_\pm \in \mathbb{Q} - \mathbb{Z}$, and are denoted by μ_α. Such fields are generally believed to reside in irreducible representations. A generic correlation function is then of the form

$$G = \left\langle \prod_i \Psi_{(h_i;k_i)}(z_i) \prod_j \Theta_{(h_j;k_k^+,k_j^-)}(w_j) \prod_l \mu_{\alpha_l}(u_l) \right\rangle \equiv \langle \prod_i \Psi_i \prod_j \Theta_j \prod_l \mu_l \rangle\,. \tag{18}$$

The zero mode contents implies now that $G = 0$ unless all three conditions

$$\mathbb{Z}_+ \ni Z_0(G) = \sum_i Z_0(\Psi_i) + \sum_j Z_0(\Theta_j) + \sum_l Z_0(\mu_l) \geq 2(r-1)\,, \tag{19}$$

$$\sum_j Z_+(\Theta_j) = \sum_j Z_-(\Theta_j) \; \in \mathbb{Z}\,, \tag{20}$$

$$\sum_l Z_+(\mu_l) \; \in \; \mathbb{Z} \quad \text{and} \quad \sum_l Z_-(\mu_l) \; \in \; \mathbb{Z} \tag{21}$$

are satisfied. These are very powerfull statements since they imply further that we can relax our condition that the logarithmic partners have to be quasi-primary. In fact, the action of L_n, $n = -1, 0, 1$, in the Ward identities eq. (6) will yield new correlation functions, $L_n G = \sum_k G'_k$. If there are contributions from fields failing to be quasi-primary, then these can be neglected if the resulting correlation functions G' do not anylonger satisfy eqs (19–21). Since L_n act as derivations, only one field in the correlator is modified in each term. Thus, if the zero mode content of a non-quasi-primary term differs from the original zero mode content such that the balance is broken, it will not contribute to the correlation function G since it does not affect the Ward identities. Hence, L_n implements a BRST like structure on the complex spanned by Z_+ and Z_-, as anticipated in [17].

In summary, we have provided the general structure of correlation functions and OPEs for LCFT with arbitrary high rank Jordan cells. We found strong constraints for correlation functions to be non-zero, intimately linked to the zero mode content of the involved fields. It remains an open problem, however, what the modular properties of such higher rank LCFTs are. These are only knonw in the rank two case [2].

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